A Computationally Efficient Algorithm for Solving Fuzzy Quadratic Programming Problems

Sumati Mahajan and S. K. Gupta

Abstract—**Quadratic programming with fuzzy parameters, an extended version of conventional quadratic programming deems fit to tackle imprecise parameters and non linear objective function. The optimum of such type of objective functions is not unique due to flexible nature of modelling parameters, rather it varies between two values. The current study proposes an alternate solution methodology coupled with (α, r) cut without using duality to deal with one of the two bi-level subprograms that handles opposite direction optimization and finds the value of the objective function. Comparative analysis has been drawn to show the simple execution and computational efficiency due to significant reduction in the number of variables, constraints and hence, the processing time. In addition, the study extends existing literature by allowing different type of cuts for the objective function and the constraints. The numerical examples are illustrated to highlight the ease and efficiency of the solution methodology.**

Index Terms—**Fuzzy parameters, Quadratic programming problem, Convex optimization, (α, r) cut.**

I. Introduction

QUADRATIC programming with crisp parameters limits its vast scope, keeping in view the rigidity involved in data collection. Instead, imprecise parameters are usually available for formulation of a model in real life scenario. Development of an efficient algorithm to find an acceptable solution for such an unstable model with interval or fuzzy parameters which is applicable in general is one of the most sought after techniques. Among frequently used methodologies are the ones which use ranking function, membership approach and duality. In fact many environment related issues e.g. water resources management, power management, noise control, flood diversion, irrigation water allocation etc have been handled using imprecise variables in linear programming, quadratic programming, dynamic programming, interval mathematical programming, fuzzy mathematical programming and stochastic mathematical programming.

The concept of impreciseness in the formulation of mathematical programming has charmed a number of researchers across different fields due to close association with real life models. The use of interval parameters were among the early efforts for the inclusion of uncertainty. Later on, the focus was shifted to fuzzy parameters which extended the notion of interval with the help of membership functions. To enrich the literature with imprecise parameters, Allahviranloo and Ghanbari [2] proposed algebraic solution of fuzzy linear systems based on interval theory. Lu et al. [3] utilised interval parameters to provide electric power system management with inexact programming approach. Safi and Razmjoo [4] investigated transportation problem with interval parameters. Li et al. [5] developed a quadratic programming model to study waste management with interval parameters. Figueroa-García et al. [6] provided optimal solutions of group fuzzy matrix games using interval valued fuzzy numbers.

Earlier also, quadratic programming problem (QPP) with fuzzy parameters has been investigated. Duality concept introduced by Dorn [7] along with membership function approach was used by Liu [8] to reduce fuzzy quadratic programming problem (FQPP) with fuzzy parameters into a pair of conventional mathematical programs and then the bounds of the objective function were found. Kheirfam and Verdegay [9] explored sensitivity analysis on FQPP. Silva et al. [10] proposed an algorithm to solve FQPP with fuzziness in the cost function by converting it into parametric multiobjective QPP. Later on, Zhou et al. [11] provided optimality conditions to solve FQPP with trapezoidal fuzzy numbers using ranking function and duality. Recently, Mirmohseni and Nasseri [12] presented a numerical method to solve FQPP with triangular fuzzy numbers in constraint coefficients. Fuzzy quadratic programming with interval numbers was also discussed by Kumar and Jeyalakshmi [13] using Simplex method and α-cut.

The objective of the present study is to provide a solution methodology for quadratic programming problems with fuzzy parameters for convex optimization type of problems. The proposed method does not use duality to find the highest value of the objecvtive function. Moreover an equivalent simplified approach is proposed. As a result, we are able to significantly reduce the number of variables to n and the number of constraints to $m+n$. The advantage of this model over the previous ones is that it is computationally efficient and significantly useful for big data problems. It is quite easy to apply due to decrease in complexity as well as decrease in the number of variables and constraints. Moreover, an (α, r)-cut provides liberty to use different type of cuts for objecvtive function and constraints. The paper is organized as such that Sect. II deals with the definition and notations, Sect. III gives the details of the methodology, Sect. IV provides illustrative examples to highlight the solution methodology and Sect. V ends up with the conclusions and future scope.

Manuscript received June 03, 2019

Sumati Mahajan and S. K. Gupta are with the Department of Mathematics, Indian Institute of Tecnology, Roorkee, Uttrakhand, 247667, India
(e-mails: sumatimahajan@pec.ac.in, skgiitr@gmail.com).

II. Preliminaries

Definition 2.1 [1] If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs: $\{(x, \mu_{\tilde{A}}(x)) | x \in X\}, \mu_{\tilde{A}}(x)$ is called the membership function of x in $\tilde{A}$ that maps X to the membership space $M = [0, 1]$.

Definition 2.2 [1] A triangular fuzzy number (TFN) $\tilde{A} = (x_1, x_1', x_1'')$ is a fuzzy set if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - x_1}{x_1' - x_1}, & x_1 < x \leq x_1' \\ \dfrac{x_1'' - x}{x_1'' - x_1'}, & x_1' \leq x < x_1'' \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.3 [1] A TFN $\tilde{A} = (x_1, x_1', x_1'')$ is called non-negative *iff* $x_1 \geq 0$.

Definition 2.4 Let $A = [\tilde{a}_{ij}]_{n\times n} = (a_{ij}, a_{ij}', a_{ij}'')_{n\times n}$ be a symmetric triangular fuzzy number matrix. Let $B = [a_{ij}]_{n\times n}, C = [a_{ij}']_{n\times n}$ and $D = [a_{ij}'']_{n\times n}$ be the corresponding crisp matrices (obtained by using lower, middle and upper entries of each of the fuzzy number entries of the matrix $[\tilde{a}_{ij}]_{n\times n}$) then the matrix $[\tilde{a}_{ij}]_{n\times n}$ is positive definite/ positive semidefinite/ negative definite/ negative semidefinite/ indefinite in accordance with all of B, C and D being positive definite/ positive semidefinite/ negative definite/ negative semidefinite/ indefinite, respectively.

Definition 2.5 [1] The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in (0, 1]$ is called the α-cut of $\tilde{A}$ and is defined as:

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}.$$

If $\tilde{A} = (x_1, x_1', x_1''), A_\alpha = [x_1 + \alpha(x_1' - x_1), x_1'' - \alpha(x_1'' - x_1')]$.

Arithmetic operations

Let $\tilde{X}_1 = (x_1, x_1', x_1'')$ and $\tilde{X}_2 = (x_2, x_2', x_2'')$ be two triangular fuzzy numbers, then

(i) $\tilde{X}_1 \oplus \tilde{X}_2 = (x_1 + x_2, x_1' + x_2', x_1'' + x_2'')$

(ii) For $k \in \mathbb{R}$, $k\tilde{X} = \begin{cases} (kx, kx', kx''), & k \geq 0, \\ (kx'', kx', kx), & k < 0. \end{cases}$

(iii) $\tilde{X}_1 \ominus \tilde{X}_2 = (x_1 - x_2'', x_1' - x_2', x_1'' - x_2)$

(iv) $\tilde{X}_1 \otimes \tilde{X}_2 \approx (p_1, p_2, p_3)$ where

$p_1 = \min \{x_1x_2, x_1x_2'', x_1''x_2, x_1''x_2''\}$,
$p_2 = \{x_1'x_2'\}$,
$p_3 = \max \{x_1x_2, x_1x_2'', x_1''x_2, x_1''x_2''\}$.

(v) If $\tilde{X}_1$ is a triangular fuzzy number and $\tilde{X}_2$ is a non-negative triangular fuzzy number, then

$$\tilde{X}_1 \otimes \tilde{X}_2 \approx \begin{cases} (x_1x_2, x_1'x_2', x_1''x_2''); x_1 \geq 0 \\ (x_1x_2'', x_1'x_2', x_1''x_2''); x_1 < 0, x_1'' \geq 0 \\ (x_1x_2'', x_1'x_2', x_1''x_2); x_1'' < 0 \end{cases}$$

III. Formulation of a fuzzy Quadratic Programming problem

The fuzzy quadratic programming problem can be formulated as:

$$\left.\begin{aligned} \text{Minimize } \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{q}_{ij} x_i x_j \\ \text{subject to } \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \ x_j \geq 0, \\ i = 1, 2, ..., m, \ j = 1, 2, ..., n. \end{aligned}\right\} \quad (1)$$

where $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$ and $\tilde{q}_{ij}$ are assumed to be fuzzy numbers and matrix $[\tilde{q}_{ij}]_{n\times n}$ is positive semi definite .

Using α-cut for the objective function and r-cut for the constraints, $\alpha, r \in (0, 1]$, the model (1) can be rewritten as :
Minimize $Z_{(\alpha,r)} =$

$$\sum_{j=1}^{n}[(c_j)_\alpha^L, (c_j)_\alpha^U]x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(q_{ij})_\alpha^L, (q_{ij})_\alpha^U]x_i x_j$$

subject to $\sum_{j=1}^{n}[(a_{ij})_r^L, (a_{ij})_r^U]x_j \leq [(b_i)_r^L, (b_i)_r^U]$,
$x_j \geq 0, i = 1, 2, ..., m, \ j = 1, 2, ..., n.$

Assume $f_{(\alpha,r)}^U, f_{(\alpha,r)}^L$ as the upper and lower bounds of the objective function respectively after applying (α, r)-cut, $\alpha, r \in (0, 1]$ on the objective function and constraints respectively, the model (1) can be divided into the following two-level IQP models as :

$$\left.\begin{aligned} f_{(\alpha,r)}^L = \min_S \left(\min_x f = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j \right) \\ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ x_j \geq 0, \ i = 1, 2, ..., m \end{aligned}\right\} \quad (2)$$

and

$$\left.\begin{aligned} f_{(\alpha,r)}^U = \max_S \left(\min_x f = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j \right) \\ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ x_j \geq 0, \ i = 1, 2, ..., m \end{aligned}\right\} \quad (3)$$

where $S = \{c_j \in [(c_j)_\alpha^L, (c_j)_\alpha^U], \ q_{ij} \in [(q_{ij})_\alpha^L, (q_{ij})_\alpha^U], a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U], b_i \in [(b_i)_r^L, (b_i)_r^U]\}$

or

$$\left.\begin{aligned} f_{(\alpha,r)}^L = \min_{S'} \left(\min_x f = \sum_{j=1}^{n} (c_j)_\alpha^L x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (q_{ij})_\alpha^L x_i x_j \right) \\ \text{subject to} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0, \ i = 1, 2, ..., m \end{aligned}\right\} \quad (4)$$

and

$$\left.\begin{aligned} f_{(\alpha,r)}^U = \max_{S'} \left(\min_x f = \sum_{j=1}^{n} (c_j)_\alpha^U x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (q_{ij})_\alpha^U x_i x_j \right) \\ \text{subject to} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, x_j \geq 0, \ i = 1, 2, ..., m \end{aligned}\right\} \quad (5)$$

where $S' = \{a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U],\ b_i \in [(b_i)_r^L, (b_i)_r^U]\}$

The problem is now to assign appropriate values to the set S' to find $f^U_{(\alpha,r)}$ and $f^L_{(\alpha,r)}$, which is decided as under:

Lower bound

Model (4) corresponds to the lower bound of the objective function of model (1) . As both the inner and outer programs have the same minimization operation, they can be combined into single programming model :

$$f^L_{(\alpha,r)} = \min_{S',x} \Bigg(\sum_{j=1}^{n} (c_j)_\alpha^L x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^L x_i x_j$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij}x_j \le b_i \Bigg),\ x_j \ge 0,\ i = 1,2,...,m$$

where $S' = \{a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U],\ b_i \in [(b_i)_r^L, (b_i)_r^U]\}$

or

$$\left.\begin{aligned} f^L_{(\alpha,r)} &= \min_{x} \Bigg(\sum_{j=1}^{n} (c_j)_\alpha^L x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^L x_i x_j \Bigg) \\ \text{subject to} &\sum_{j=1}^{n} (a_{ij})_r^L x_j \le (b_i)_r^U, x_j \ge 0,\ i = 1,2,...,m. \end{aligned}\right\} \quad (6)$$

(as the maximum possible region is determined by $(a_{ij})_r^L$ and $(b_i)_r^U$).

A. The proposed result:

It will be shown that $f^U_{(\alpha,r)}$, the upper bound of the objective function can be found without using duality and hence, drastically curtails the number of variables, constraints and processing time.

Upper bound

Model (5) corresponds to the upper bound of the objective function of model (1) , but as the optimization is in different directions, the direction of the inner is also changed to maximization using duality as follows

Duality approach

The Lagrangian dual formulation of the problem corresponding to highest value is to maximize $\theta(\lambda, \delta)$, which is given by

$$\theta(\lambda,\delta) = \inf \Bigg\{ \sum_{j=1}^{n} (c_j)_\alpha^U x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j + \sum_{i=1}^{m}\lambda_i \Bigg(\sum_{j=1}^{n} a_{ij}x_j - b_i \Bigg) + \sum_{j=1}^{n}\delta_j x_j \Bigg\}$$

where $\lambda_i, \delta_j, x_j \ge 0$ and $a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U]$, $b_i \in [(b_i)_r^L, (b_i)_r^U],\ \forall i,\ j.$

Here, the function $\theta(\lambda, \delta)$ is a convex function as $[\tilde{q}_{ij}]_{n\times n}$ is a symmetric positive semidefinite matrix. The necessary and sufficient condition for a solution to attain maxima is that gradient of $\theta(\lambda, \delta)$ should vanish.

Hence, the inner level model in the problem corresponding to highest value transforms to

$$\max_{x,\lambda,\delta} \Bigg(\sum_{j=1}^{n} (c_j)_\alpha^U x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j + \sum_{i=1}^{m}\lambda_i \Bigg(\sum_{j=1}^{n} a_{ij}x_j - b_i \Bigg) - \sum_{j=1}^{n}\delta_j x_j \Bigg)$$

subject to

$$(c_j)_\alpha^U + \sum_{i=1}^{n}(q_{ij})_\alpha^U x_i + \sum_{i=1}^{m}\lambda_i a_{ij} - \delta_j = 0,\ j = 1,2,...,n,$$

where $a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U], b_i \in [(b_i)_r^L, (b_i)_r^U]$,
$\lambda_i, \delta_j, x_j \ge 0,\ i = 1,2,...,m,\ j = 1,2,...,n.$

In view of the duality concept, the problem to find highest value becomes:

$$(f)^U_{\alpha,r} = \max_{S',x,\lambda,\delta} \Bigg(\sum_{j=1}^{n} (c_j)_\alpha^U x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j + \sum_{i=1}^{m}\lambda_i \big(\sum_{j=1}^{n} a_{ij}x_j - b_i \big) - \sum_{j=1}^{n}\delta_j x_j \Bigg)$$

subject to

$$(c_j)_\alpha^U + \sum_{i=1}^{n}(q_{ij})_\alpha^U x_i + \sum_{i=1}^{m}\lambda_i a_{ij} - \delta_j = 0,\ j = 1,2,...,n,$$

$\lambda_i, \delta_j, x_j \ge 0,\ i = 1,2,...,m,\ j = 1,2,...,n$
where $S' = \{(a_{ij}, b_i) : a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U], b_i \in [(b_i)_r^L, (b_i)_r^U],\ \forall\ i, j\}$.

Since $(c_j)_\alpha^U + \sum_{i=1}^{n}(q_{ij})_\alpha^U x_i + \sum_{i=1}^{m}\lambda_i a_{ij} - \delta_j = 0$, $j = 1,2,...,n$, therefore

$$\sum_{j=1}^{n} (c_j)_\alpha^U x_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_i a_{ij}x_j - \sum_{j=1}^{n}\delta_j x_j = -\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j,\ j = 1,2,...,n.$$

Thus, the above model reduces to

$$(f)^U_{\alpha,r} = \max_{S',x,\lambda,\delta} \Bigg(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j - \sum_{i=1}^{m}\lambda_i b_i \Bigg)$$

subject to

$$\sum_{i=1}^{n}(q_{ij})_\alpha^U x_i + \sum_{i=1}^{m}\lambda_i a_{ij} - \delta_j = -(c_j)_\alpha^U,\ j = 1,2,...,n,$$

$\lambda_i, \delta_j, x_j \ge 0,\ i = 1,2,...,m,\ j = 1,2,...,n$
where
$S' = \{\ a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U],\ b_i \in [(b_i)_r^U, (b_i)_r^U], \forall\ i, j\}$.

Further, as $(b_i)_r^L \le b_i \le (b_i)_r^U$ and $\lambda_i \ge 0$ for all i, it follows that

$$(f)^U_{\alpha,r} = \max_{S'',x,\lambda,\delta} \Bigg(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})_\alpha^U x_i x_j - \sum_{i=1}^{m}\lambda_i (b_i)_r^L \Bigg)$$

subject to

$$\sum_{i=1}^{n}(q_{ij})_\alpha^U x_i + \sum_{i=1}^{m}\lambda_i a_{ij} - \delta_j = -(c_j)_\alpha^U,\ j = 1,2,...,n,$$

$\lambda_i, \delta_j, x_j \ge 0,\ i = 1,2,...,m,\ j = 1,2,...,n$
where $S'' = \{a_{ij} : a_{ij} \in [(a_{ij})_r^L, (a_{ij})_r^U], \forall\ i, j\}$.

Finally, since $(a_{ij})_r^L \le a_{ij} \le (a_{ij})_r^U$ for all i and j, therefore it yields

$$\left.\begin{array}{r}(f)^U_{\alpha,r} = \max_{x,\lambda,\delta}\Big(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j - \sum_{i=1}^{m}\lambda_i(b_i)^L_r\Big)\\ \text{subject to } \sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^L_r - \delta_j \le -(c_j)^U_\alpha,\\ \sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^U_r - \delta_j \ge -(c_j)^U_\alpha,\\ \lambda_i,\delta_j,x_j \ge 0,\ i=1,2,...,m,\ j=1,2,...,n.\end{array}\right\} \quad (7)$$

Proposed method

We now propose a modified approach to find upper bound $f^U_{(\alpha,r)}$ of the problem (1). The new formulation involves considerably lesser number of constraints and variables as compared to problem (5) and hence an efficient approach. In addition the new formulation is applicable for concave type of optimization also.

We claim that the highest value $f^U_{(\alpha,r)}$ can be found by simply solving the following optimization model:

$$\left.\begin{array}{r}f^U_{(\alpha,r)} = \min_{x}\Big(\sum_{j=1}^{n}(c_j)^U_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j\Big)\\ \text{subject to } \sum_{j=1}^{n}(a_{ij})^U_r x_j \le (b_i)^L_r,\ i=1,2,...,m,\\ x_j \ge 0,\ j=1,2,...,n.\end{array}\right\} \quad (8)$$

Proof: Since $(a_{ij})^L_r \le (a_{ij})^U_r$, therefore the above problem is equivalent to

$$\left.\begin{array}{r}f^U_{(\alpha,r)} = \min_{x}\Big(\sum_{j=1}^{n}(c_j)^U_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j\Big)\\ \text{subject to } \sum_{j=1}^{n}(a_{ij})^U_r x_j \le (b_i)^L_r,\ i=1,2,...,m,\\ \sum_{j=1}^{n}(a_{ij})^L_r x_j \le (b_i)^L_r,\ i=1,2,...,m,\\ x_j \ge 0,\ j=1,2,3...,n.\end{array}\right\} \quad (9)$$

Now, we will show that the dual model of problem (9) is identical to (7). Let λ_i, μ_i and δ_j be the Lagrange's multipliers to the constraints of the above problem in that order, then the dual of the problem will be:

$$\max_{x,\lambda,\mu,\delta}\Big(\sum_{j=1}^{n}(c_j)^U_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j + \sum_{i=1}^{m}\lambda_i\big(\sum_{j=1}^{n}(a_{ij})^U_r x_j - (b_i)^L_r\big) + \sum_{i=1}^{m}\mu_i\big(\sum_{j=1}^{n}(a_{ij})^L_r x_j - (b_i)^L_r\big) - \sum_{j=1}^{n}\delta_j x_j\Big)$$

subject to

$$(c_j)^U_\alpha + \sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^U_r + \sum_{i=1}^{m}\mu_i(a_{ij})^L_r - \delta_j = 0,\ j=1,2,...,n,$$

$\lambda_i,\mu_i,\delta_j,x_j \ge 0\ \forall\ i$ and j.

The equation $(c_j)^U_\alpha + \sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^U_r + \sum_{i=1}^{m}\mu_i(a_{ij})^L_r - \delta_j = 0$, for $j=1,2,...,n$, implies

$$\sum_{j=1}^{n}(c_j)^U_\alpha x_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_i(a_{ij})^U_r x_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i(a_{ij})^L_r x_j - \sum_{j=1}^{n}\delta_j x_j = -\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j,\ j=1,2,...,n.$$

It follows that

$$f^U_{(\alpha,r)} = \max_{x,\lambda,\mu,\delta}\Big(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j - \sum_{i=1}^{m}\lambda_i(b_i)^L_r - \sum_{i=1}^{m}\mu_i(b_i)^L_r\Big)$$

subject to

$$\sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^U_r + \sum_{i=1}^{m}\mu_i(a_{ij})^L_r - \delta_j = -(c_j)^U_\alpha,\ j=1,2,...,n,$$

$$\lambda_i,\delta_j,x_j \ge 0,\ i=1,2,...,m,\ j=1,2,...,n.$$

Further, using $(a_{ij})^L_r \le a_{ij} \le (a_{ij})^U_r$, we get

$$f^U_{(\alpha,r)} = \max_{x,\lambda,\mu,\delta}\Big(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(q_{ij})^U_\alpha x_i x_j - \sum_{i=1}^{m}\lambda_i(b_i)^L_r - \sum_{i=1}^{m}\mu_i(b_i)^L_r\Big)$$

subject to

$$\sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^L_r + \sum_{i=1}^{m}\mu_i(a_{ij})^L_r - \delta_j \le -(c_j)^U_\alpha,$$
$$\sum_{i=1}^{n}(q_{ij})^U_\alpha x_i + \sum_{i=1}^{m}\lambda_i(a_{ij})^U_r + \sum_{i=1}^{m}\mu_i(a_{ij})^U_r - \delta_j \ge -(c_j)^U_\alpha,$$
$$\lambda_i,\mu_i,\delta_j \ge 0,\ i=1,2,...,m,\ j=1,2,...,n.$$

Finally, replacing $\lambda_i + \mu_i$ by $\nu_i,\ i=1,2,...,m$ in the above, we get model (7). This proves our claim. □

Remark 1 On the same lines, we get similar results in case of a maximization problem. In particular, the lowest and the highest values of the following model

$$\left.\begin{array}{r}\text{Maximize } \tilde{Z} = \sum_{j=1}^{n}\tilde{c}_j x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{q}_{ij}x_i x_j\\ \text{subject to } \sum_{j=1}^{n}\tilde{a}_{ij}x_j \ge \tilde{b}_i,\ x_j \ge 0,\\ i=1,2,...,m,\ j=1,2,...,n.\end{array}\right\} \quad (10)$$

where $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$ and $\tilde{q}_{ij}$ are assumed to be fuzzy numbers and matrix $[\tilde{q}_{ij}]_{n\times n}$ is negative semi definite,

can be found by solving

ISBN: 978-988-14048-7-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\left.\begin{aligned} f^L_{(\alpha,r)} &= \max_x \left(\sum_{j=1}^{n} (c_j)^L_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (q_{ij})^L_\alpha x_i x_j \right) \\ \text{subject to} &\sum_{j=1}^{n} (a_{ij})^L_r x_j \geq (b_i)^U_r, x_j \geq 0,\ i = 1,2,...,m. \end{aligned}\right\} \quad (11)$$

and

$$\left.\begin{aligned} f^U_{(\alpha,r)} &= \max_x \left(\sum_{j=1}^{n} (c_j)^U_\alpha x_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (q_{ij})^U_\alpha x_i x_j \right) \\ \text{subject to} &\sum_{j=1}^{n} (a_{ij})^U_r x_j \geq (b_i)^L_r, x_j \geq 0,\ i = 1,2,...,m. \end{aligned}\right\} \quad (12)$$

TABLE I
COMPARISON OF PROPOSED METHOD WITH LIU'S METHOD

	No. of variables	No. of constraints	Comput. efficient	Ease of application	Flex. in cuts
Liu's method	$mn + m + 2n$	$2mn + 2m + n$	×	×	×
Proposed method	n	$m + n$	✓	✓	✓

As a result the membership of the objective function can be put as:

$$\mu_{\tilde{Z}}(x) = \begin{cases} L(Z), & f^L_{(0,0)} < Z \leq f^L_{(1,0)}, \\ 1, & f^L_{(1,0)} < Z \leq f^U_{(1,1)}, \\ R(Z), & f^U_{(1,1)} < Z \leq f^U_{(0,1)}, \\ 0, & \text{otherwise} \end{cases}$$

where $L(Z)$ and $R(Z)$ are left and right shape functions respectively of $\mu_{\tilde{Z}}(x)$.

Advantages of the proposed method over existing methods

1) The proposed method contributes in solving a fuzzy quadratic programming in an efficient way due to significant reduction in the variables and constraints.
2) The proposed method does not use duality as done by Liu [8] and thus is vey much simple to handle and apply.
3) As objective function and constraints are of entirely different nature, reasonably different type of cuts are suggested. Here α-cut for objective function and r-cut for constraints is proposed.

IV. Illustrative Examples :

Example 4.1 Let's take the example considered by Liu [8]

Minimize $f = (-6,-5,-4)x_1 + (1,1.5,2)x_2 + (-3,-2,-1)x_1x_2 + (2,3,4)x_1^2 + (1,2,3)x_2^2$
subject to

$$x_1 + (0.5,1,1.5)x_2 \leq (1,2,3),$$

$$(1,2,3)x_1 + (-2,-1,-0.5)x_2 \leq (3,4,5), \quad x_1 \geq 0, x_2 \geq 0.$$

Solution : For $\alpha, r \in (0,1]$, the model (6) gives,

$$f^L_{(\alpha,r)} = \min_x \big((-6+\alpha)x_1 + (1+0.5\alpha)x_2 + (-3+\alpha)x_1x_2 + (2+\alpha)x_1^2 + (1+\alpha)x_2^2\big)$$

subject to

$$x_1 + (0.5+.5r)x_2 \leq (3-r)$$

$$(1+r)x_1 + (-2+r)x_2 \leq (5-r),\ x_1,x_2 \geq 0$$

and from model (8), we get

$$f^U_{(\alpha,r)} = \min_x \big((-4-\alpha)x_1 + (2-0.5\alpha)x_2 + (-1-\alpha)x_1x_2 + (4-\alpha)x_1^2 + (3-\alpha)x_2^2\big)$$

subject to

$$x_1 + (1.5-0.5r)x_2 \leq (1+r)$$

$$(3-r)x_1 + (-0.5-0.5r)x_2 \leq (3+r),\ x_1,x_2 \geq 0$$

The result is summed up as under in Table II.

TABLE II
VALUE OF $f = [f^L_{\alpha,r}, f^U_{\alpha,r}]$ AT DIFFERENT (α, r)-CUTS

$\alpha\backslash r$	0.0*	0.3	0.7	1.0
0.0*	[−10.08, −1.00]	[−8.50, −1.00]	[−6.94, −1.00]	[−6.04, −1.00]
0.2	[−7.20, −1.16]	[−6.48, −1.16]	[−5.63, −1.16]	[−5.07, −1.16]
0.4	[−4.47, −1.34]	[−4.47, −1.34]	[−4.32, −1.34]	[−4.10, −1.34]
0.6	[−3.14, −1.56]	[−3.14, −1.56]	[−3.14, −1.56]	[−3.14, −1.56]
0.8	[−2.49, −1.80]	[−2.49, −1.80]	[−2.49, −1.80]	[−2.49, −1.80]
1.0	[−2.09, −2.09]	[−2.09, −2.09]	[−2.09, −2.09]	[−2.09, −2.09]

Example 4.2 Consider another example as below:

Maximize $f = (6,7,8)x_1 + (-4,-3,-2)x_2 + (2,4,6)x_1x_2 + (-7,-5,-4)x_1^2 + (-8,-6,-4)x_2^2$
subject to

$$(0,1,2)x_1 + (1,2,3)x_2 \geq (5,7,9),$$

$$(2,4,6)x_1 + (-4,-2,-1)x_2 \geq (4,5,6), \quad x_1 \geq 0, x_2 \geq 0.$$

Solution : For $\alpha, r \in (0,1]$, we get the lowest value,

$$f^L_{(\alpha,r)} = \min_x \big((6+\alpha)x_1 + (-4+\alpha)x_2 + (2+2\alpha)x_1x_2 + (-7+2\alpha)x_1^2 + (-8+2\alpha)x_2^2\big)$$

subject to

$$(r)x_1 + (1+r)x_2 \geq (9-2r)$$

$$(2+2r)x_1 + (-4+2r)x_2 \geq (6-r),\ x_1,x_2 \geq 0$$

and the highest value as

$$f^U_{(\alpha,r)} = \min_x \big((8-\alpha)x_1 + (-2-\alpha)x_2 + (6-2\alpha)x_1x_2 + (-4-\alpha)x_1^2 + (-4-2\alpha)x_2^2\big)$$

subject to

$$(2-r)x_1 + (3-r)x_2 \geq (5+2r)$$

$$(6-2r)x_1 + (-1-r)x_2 \geq (4+r),\ x_1,x_2 \geq 0$$

The result is summed up as under in Table III.

TABLE III
VALUE OF $f = [f^L_{\alpha,r}, f^U_{\alpha,r}]$ AT DIFFERENT (α, r)-CUTS

$\alpha \backslash r$	0.0^*	0.3	0.7	1.0
0.0^*	$[-3267, 4.04]$	$[-529.7, 4.74]$	$[-135.0, 5.77]$	$[-66.4, 6.28]$
0.2	$[-2977, 3.59]$	$[-477.1, 4.03]$	$[-120.2, 4.48]$	$[-58.8, 4.51]$
0.4	$[-2686, 3.22]$	$[-424.5, 3.50]$	$[-105.3, 3.62]$	$[-51.3, 3.62]$
0.6	$[-2396, 2.93]$	$[-371.9, 3.06]$	$[-90.4, 3.08]$	$[-43.7, 3.08]$
0.8	$[-2105, 2.67]$	$[-319.3, 2.71]$	$[-75.5, 2.71]$	$[-36.1, 2.71]$
1.0	$[-1815, 2.44]$	$[-266.6, 2.45]$	$[-60.7, 2.45]$	$[-28.4, 2.45]$

V. CONCLUSION

The present study suggests a computationally efficient alternate approach to investigate fuzzy quadratic programming without using duality. As a result, significant number of variables and constraints are reduced. Consequently, it helps in saving processing time as well. This approach will definitely go a long way to simplify the handling process of fuzziness in mathematical programming especially in big data problems. Moreover usual α-cut need not simultaneously govern the objective function and the constraints, so (α, r)-cut is proposed for fuzzy quadratic programming. The results presented by Liu [8] are achieved when $\alpha = r$ and present a subset of the proposed approach. In future, the approach can be extended to other nonlinear programming problems.

ACKNOWLEDGMENT

The first author thankfully acknowledges the Quality Improvement Programme cell of IIT Roorkee and Punjab Engineering College, Chandigarh along with All India Council of Technical Education for their support to carry out this work.

REFERENCES

[1] H.J. Zimmermann, "Fuzzy set theory and its applications", Boston, Kluwer-Nijhoff, 1996.
[2] T. Allahviranloo and M. Ghanbari, "On the algebraic solution of fuzzy linear systems based on interval theory", *Applied Mathematical Modelling*, vol. 36, no. 11, pp. 5360-5379, 2012.
[3] H. W. Lu, M. F. Cao, J. Li, G. H. Huang and L. He, "An inexact programming approach for urban electric power systems management under random-interval-parameter uncertainty", *Applied Mathematical Modelling*, vol.39, no. 7, pp.1757-1768, 2015.
[4] M. R. Safi and A. Razmjoo, "Solving fixed charge transportation problem with interval parameters", *Applied Mathematical Modelling*, vol.37, no. 18-19, pp. 8341-8347, 2013.
[5] Y. P. Li, G. H. Huang and S. L. Nie, "A mathematical model for identifying an optimal waste management policy under uncertainty", *Applied Mathematical Modelling*, vol. 36, no. 6, pp. 2658-2673, 2012.
[6] J. C. Figueroa-García, A. Mehra and S. Chandra, "Optimal solutions for group matrix games involving interval-valued Fuzzy Numbers", *Fuzzy Sets and Systems*, vol. 362, pp. 55-70,(2019.
[7] W S. Dorn, " Duality in quadratic programming", *Quarterly of Applied Mathematics*, vol. 18, no. 2, pp. 155-162, 1960.
[8] S.T. Liu, "A revisit to quadratic programming with fuzzy parameters", *Chaos, Solitons and Fractals*, vol.41, pp. 1401-1407, 2009.
[9] B. Kheirfam and J.L Verdegay, "Strict sensitivity analysis in fuzzy quadratic programming", *Fuzzy Sets and Systems*, vol. 198, pp. 99-111, 2012.
[10] R. C. Silva, C. Cruz and J.L. Vedegay, "Fuzzy costs in quadratic programming problems", *Fuzzy Optimization and Decision Making*, vol 12, pp. 231-248, 2013.
[11] X. G. Zhou, B. Y. Cao and S. H. Nasseri, "Optimality conditions for fuzzy number quadratic programming with fuzzy coefficients", *Journal of Applied Mathematics*, pp. 1-8, 2014.
[12] S. M. Mirmohseni and S.H. Nasseri, "A Quadratic programming with Triangular Fuzzy numbers", *Journal of Applied Mathematics and Physics*, vol. 5, pp. 2218-2227, 2017.
[13] C. A. Kumar and R. R. Jeyalakshmi, "The maximum index method for solving the fuzzy quadratic programming problem with interval numbers using different levels of α-cuts", *Journal of Engineering Technology*, vol. 6, pp. 240-247, 2017.

Date of modification : September 5, 2019.
Brief description of the changes : Compatibility of citation of the references.